Hausdorff Continuous Solutions of Nonlinear PDEs through the Order Completion Method

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Abstract

It was shown in [13] that very large classes of nonlinear PDEs have solutions which can be assimilated with usual measurable functions on the Euclidean domains of definition of the respective equations. In this paper the regularity of these solutions has significantly been improved by showing that they can in fact be assimilated with Hausdorff continuous functions. The method of solution of PDEs is based on the Dedekind order completion of spaces of smooth functions which are defined on the domains of the given equations.

1 Introduction

The following significant *threefold* breakthrough was obtained in [13] with respect to solving large classes of nonlinear PDEs, see MR 95k:35002. Namely:

a) arbitrary nonlinear PDEs of the form

$$T(x, D)u = f(x), \quad x \in \Omega, \tag{1}$$

where

$$T(x, D)u = g(x, u(x), \ \ldots, D_x^p u(x), \ \ldots), \quad p \in \mathbb{N}^n, \ |p| \leq m, \tag{2}$$

with g jointly continuous in all it arguments, f in a class of measurable functions, $\Omega \subseteq \mathbf{R}^n$ arbitrary open, $m \in \mathbb{N}$ arbitrary given, and the unknown function $u : \Omega \longrightarrow \mathbb{R}$, were proven to have

b) solutions u which can be assimilated with usual measurable functions on Ω, and

c) the solution method was based on the Dedekind order completion of suitable spaces of smooth functions on Ω.

In fact, the conditions at a) can further be relaxed by assuming that g may admit certain *discontinuities*, namely, that it is continuous only on $(\Omega \backslash \Sigma) \times \mathbf{R}^{m^*}$, where Σ is a closed, nowhere dense subset of Ω, while m^* is the number of arguments in g minus n. This relaxation on the continuity of g may be significant since such subsets of discontinuity Σ can have arbitrary large positive Lebesgue measure.

In this way, the solutions of the unprecedented large class of nonlinear PDEs in (1) can be obtained *without* the use of any sort of distributions, hyperfunctions, generalized functions, or of methods of functional analysis. Moreover, one obtains a general, *blanket regularity*, given by the fact that the solutions constructed can be assimilated with usual measurable functions on the corresponding domains Ω in Euclidean spaces.

In this paper we discuss a further significant improvement of the above mentioned results with respect to the *regularity* properties of the solutions. Namely, this time we show that they can be assimilated with the significantly smaller class of Hausdorff continuous functions on the open domains Ω. This improvement follows, among others, from a recent breakthrough, see [1], which solves a long outstanding problem related to the Dedekind order completion of spaces $C(X)$ of real valued continuous functions on rather arbitrary topological spaces X.

The Hausdorff continuous functions are not unlike the usual real valued continuous functions. For instance, they assume real values on a dense subset of the domain and are completely determined by the values on this subset. However, these functions may also assume interval values on a certain subset of the domain. Hence the concept of Hausdorff continuity is formulated within the realm of interval valued functions. We denote by $\mathbb{A}(\Omega)$ the set of all functions defined on an open set $\Omega \subset \mathbb{R}^n$ with values which are finite or infinite closed real intervals, that is,

$$\mathbb{A}(\Omega) = \{f : \Omega \to \mathbb{I}\overline{\mathbb{R}}\},$$

where $\mathbb{I}\overline{\mathbb{R}} = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\},\ \underline{a} \leq \overline{a}\}$. Given an interval $a = [\underline{a}, \overline{a}] \in \mathbb{I}\overline{\mathbb{R}}$,

$$w(a) = \begin{cases} \overline{a} - \underline{a} & \text{if} \quad \underline{a}, \overline{a} \text{ finite}, \\ +\infty & \text{if} \quad \underline{a} < \overline{a} = +\infty \text{ or } \underline{a} = -\infty < \overline{a}, \\ 0 & \text{if} \quad \underline{a} = \overline{a} = \pm\infty, \end{cases}$$

is the width of a, while $|a| = \max\{|\underline{a}|, |\overline{a}|\}$ is the modulus of a. An extended real interval a is called proper if $w(a) > 0$ and degenerate or point if $w(a) = 0$. Identifying $a \in \overline{\mathbb{R}}$ with the degenerate interval $[a, a] \in \mathbb{I}\overline{\mathbb{R}}$, we consider $\overline{\mathbb{R}}$ as a subset of $\mathbb{I}\overline{\mathbb{R}}$. In this way $\mathbb{A}(\Omega)$ contains the set of extended real valued functions, namely,

$$\mathcal{A}(\Omega) = \{f : \Omega \to \overline{\mathbb{R}}\}.$$

A partial order which extends the total order on $\overline{\mathbb{R}}$ can be defined on $\mathbb{I}\overline{\mathbb{R}}$ in more than one way. However, it will prove useful to consider on $\mathbb{I}\overline{\mathbb{R}}$ the partial order $\leq$ defined by

$$[\underline{a}, \overline{a}] \leq [\underline{b}, \overline{b}] \Longleftrightarrow \underline{a} \leq \underline{b},\ \overline{a} \leq \overline{b}. \tag{3}$$

The partial order induced on $\mathbb{A}(\Omega)$ by (3) in a point-wise way, i.e.,

$$f \leq g \Longleftrightarrow f(x) \leq g(x),\ x \in \Omega, \tag{4}$$

is an extension of the usual point-wise order in the set of extended real valued functions $\mathcal{A}(\Omega)$.

The application of Hausdorff continuous functions to problems in Analysis, e.g. [1], and to nonlinear PDEs as in this paper, are based on the quite extraordinary fact that the set $\mathbb{H}(\Omega)$ of all Hausdorff continuous functions on Ω is order complete while some of its important subsets are Dedekind order complete. We can recall that the usual spaces of real valued functions considered in Analysis or Functional Analysis, e.g. spaces of continuous functions, Lebesgue spaces, Sobolev spaces, are with very few exceptions neither order complete nor Dedekind order complete.

The definition of the concept of Hausdorff continuity and related terminology are discussed in Section 2. The Baire operators and the graph completion operator which are instrumental for the definition and the properties of Hausdorff continuous functions are also discussed in that section. In order to improve the readability of the paper, a short account of some basic properties of the Hausdorff continuous functions is given in the Appendix.

The use of extended real intervals in the definition of the set $\mathbb{A}(\Omega)$ is partially motivated by the fact that the Baire operators involve infimums and supremums which might not exists in the realm of the usual (finite) real intervals. However, the main motivation with regard to the present exposition is the need to accommodate solutions of PDEs which are discontinuous at certain points of the domain Ω and unbounded in the neighborhood of these points, e.g. the so called finite time blow-up. For this purpose it will prove sufficient to consider only the nearly finite functions.

Definition 1 *A function $f \in \mathbb{A}(\Omega)$ is called nearly finite if there exists an open and dense subset D of Ω such that*

$$|f(x)| < +\infty,\ x \in D$$

After a brief introduction to the order completion method in Section 3 we give the main result of the paper in Section 4, namely, that the solutions of the equation (1) through the order completion method can be assimilated with nearly finite Hausdorff continuous functions.

2 Baire operators, graph completion operator and Hausdorff continuity

For every $x \in \Omega$, $B_\delta(x)$ denotes the open δ-neighborhood of x in Ω, that is,

$$B_\delta(x) = \{y \in \Omega : ||x - y|| < \delta\}.$$

Let D be a dense subset of Ω. The pair of mappings $I(D,\Omega,\cdot)$, $S(D,\Omega,\cdot)$: $\mathbb{A}(D) \to \mathcal{A}(\Omega)$ defined by

$$\begin{aligned} I(D,\Omega,f)(x) &= \sup_{\delta>0} \inf\{z \in f(y) : y \in B_\delta(x) \cap D\}, x \in \Omega, \quad (5) \\ S(D,\Omega,f)(x) &= \inf_{\delta>0} \sup\{z \in f(y) : y \in B_\delta(x) \cap D\}, x \in \Omega, \quad (6) \end{aligned}$$

are called lower Baire and upper Baire operators, respectively. Clearly for every $f \in \mathbb{A}(D)$ we have

$$I(D,\Omega,f)(x) \leq f(x) \leq S(D,\Omega,f)(x),\ x \in \Omega.$$

Hence the mapping $F : \mathbb{A}(D) \to \mathbb{A}(\Omega)$, called a graph completion operator, where

$$F(D,\Omega,f)(x) = [I(D,\Omega,f)(x), S(D,\Omega,f)(x)],\ x \in \Omega,\ f \in \mathbb{A}(\Omega), \quad (7)$$

is well defined and we have the inclusion

$$f(x) \subseteq F(f)(x),\ x \in \Omega. \quad (8)$$

The name of this operator is derived from the fact that considering the graphs of f and $F(D,\Omega,f)$ as subsets of the topological space $\Omega \times \overline{\mathbb{R}}$, the graph of $F(D,\Omega,f)$ is the minimal closed set which is a graph of interval function on Ω and contains the the graph of f. In the case when $D = \Omega$ the sets D and Ω will be usually omitted from the operators' argument lists, that is,

$$I(f) = I(\Omega,\Omega,f)\ ,\quad S(f) = S(\Omega,\Omega,f)\ ,\quad F(f) = F(\Omega,\Omega,f)$$

Let us note that, the graph completion operator is monotone about inclusion with respect to the functional argument, that is, if $f, g \in \mathbb{A}(D)$ where D is dense in Ω then

$$f(x) \subseteq g(x),\ x \in D \Longrightarrow F(D,\Omega,f)(x) \subseteq F(D,\Omega,g)(x),\ x \in \Omega. \quad (9)$$

Furthermore, the graph completion operator is monotone about inclusion with respect to the set D in the sense that if D_1 and D_2 are dense subsets of Ω and $f \in \mathbb{A}(D_1 \cup D_2)$ then

$$D_1 \subseteq D_2 \Longrightarrow F(D_1,\Omega,f)(x) \subseteq F(D_2,\Omega,f)(x), x \in \Omega. \quad (10)$$

This, in particular, means that for any dense subset D of Ω and $f \in \mathbb{A}(\Omega)$ we have

$$F(D,\Omega,f)(x) \subseteq F(f)(x), x \in \Omega. \quad (11)$$

Let $f \in \mathbb{A}(\Omega)$. For every $x \in \Omega$ the value of f is an interval $[\underline{f}(x), \overline{f}(x)]$. Hence, the function f can be written in the form $f = [\underline{f}, \overline{f}]$ where $\underline{f}, \overline{f} \in \mathcal{A}(X)$ and $\underline{f} \leq \overline{f}$. The lower and upper Baire operators as well as the graph completion

operator of an interval valued function $f=[\underline{f},\overline{f}]\in\mathcal{A}(\Omega)$ can be conveniently represented in terms of the functions $\underline{f}$ and $\overline{f}$:

$$I(D,\Omega,f)=I(D,\Omega,\underline{f})\ ,\ \ S(D,\Omega,f)=S(D,\Omega,\overline{f}),$$

$$F(D,\Omega,f)=[I(D,\Omega,\underline{f}),S(D,\Omega,\overline{f})].$$

Definition 2 *A function $f\in\mathbb{A}(\Omega)$ is called Hausdorff continuous, or H-continuous, if for every function $g\in\mathbb{A}(\Omega)$ which satisfies the inclusion $g(x)\subseteq f(x)$, $x\in\Omega$, we have $F(g)(x)=f(x)$, $x\in\Omega$.*

The concepts of Hausdorff continuity is strongly connected to the concepts of semi-continuity of real functions. We have the following characterization of the fixed points of the lower and the upper Baire operators, see [5]:

$$I(f)\quad=f\Longleftrightarrow f-\text{lower semi}-\text{continuous on }\Omega \tag{12}$$
$$S(f)\quad=f\Longleftrightarrow f-\text{upper semi}-\text{continuous on }\Omega \tag{13}$$

Hence an interval function $f=[\underline{f},\overline{f}]$ is H-continuous if and only if the following three conditions hold

(i) $\underline{f}$ is lower semi-continuous

(ii) $\overline{f}$ is upper semi-continuous

(iii) the set $\{\phi\in\mathcal{A}(\Omega):\underline{f}\leq\phi\leq\overline{f}\}$ does not contain lower or upper semi-continuous functions other than $\underline{f}$ and $\overline{f}$

The concept of H-continuity can be considered as a generalization of the concept of continuity of real functions in the sense that the only real (point valued) functions contained in $\mathbb{H}(\Omega)$ are the continuous functions, that is,

$$\left.\begin{array}{c}f\in\mathcal{A}(\Omega)\\ f\ \text{is H-continuous}\end{array}\right\}\Longrightarrow f\ \ \text{is continuous} \tag{14}$$

The H-continuous functions retain some essential properties of the usual real continuous functions as stated in Theorem 9 in the Appendix. Further links with the real continuous functions are presented in Theorems 10 and 11. We should also note that any Hausdorff continuous function f is "essentially" point valued in the sense that it assumes point values everywhere except on a set W_f which is of first Baire category, see Theorem 8. Through an application of the Baire category theorem this implies that the complement of W_f in Ω is a set of second Baire category. Hence

$$D_f=\Omega\setminus W_f=\{x\in\Omega:f(x)\in\overline{\mathbb{R}}\}\ \ \text{is dense in }\Omega. \tag{15}$$

3 Order completion method for nonlinear PDEs.

The order completion method in solving general nonlinear systems of PDEs of the form (1) is based on certain very simple, even if less than usual, approximation properties, see [13]. To give an idea about the ways the order completion method works, we mention some of these approximations here.

The differential operator $T(x, D)$ o n the left hand side of (1) has the following basic approximation property :

Lemma 3

$$\forall \quad x_0 \in \Omega, \quad \epsilon > 0 \quad :$$

$$\exists \quad \delta > 0, \quad P \quad \textit{polynomial in} \quad x \in \mathbf{R}^n \quad :$$

$$||x - x_0|| \ \leq \ \delta \quad \Longrightarrow \quad f(x) - \epsilon \ \leq \ T(x, D)P(x) \ \leq \ f(x)$$

$\square$

Consequently, we obtain :

Proposition 4

$$\forall \quad \epsilon > 0 \quad :$$

$$\exists \quad \Gamma_\epsilon \subset \Omega \quad \textit{closed, nowhere dense in} \quad \Omega, \quad U_\epsilon \in C^\infty(\Omega) \quad :$$

$$f - \epsilon \ \leq \ T(x, D)P \ \leq \ f \quad \textit{on} \quad \Omega \setminus \Gamma_\epsilon$$

Furthermore, one can also assume that the Lebesgue measure of Γ_ϵ *is zero, namely*

$$mes \ (\Gamma_\epsilon) \ = \ 0.$$

$\square$

In view of Proposition 4, the spaces of piecewise smooth functions given by

$$C^l_{nd}(\Omega) \ = \ \left\{ \ u \ \middle| \ \begin{array}{l} \exists \ \ \Gamma \subset \Omega \ \ \text{closed, nowhere dense} \ \ : \\ \\ \quad *) \ u : \Omega \setminus \Gamma \to \mathbf{R} \\ \\ \quad **) \ u \in C^l(\Omega \setminus \Gamma) \end{array} \right\} \tag{16}$$

where $l \in \mathbf{N}$, are considered. It is easy to see that we have the inclusion

$$T(x, D) \ C^m_{nd}(\Omega) \ \subseteq \ C^0_{nd}(\Omega) \tag{17}$$

The general existence result obtained in [13] is represented through the following equation

$$T(x, D)^\# \ (C^m_{nd}(\Omega))^\#_T \ = \ (C^0_{nd}(\Omega))^\# \tag{18}$$

Here $(C^m_{nd}(\Omega))^{\#}_T$ and $(C^0_{nd}(\Omega))^{\#}$ are Dedekind order completions of $C^m_{nd}(\Omega)$ and $C^0_{nd}(\Omega)$, respectively, when these latter two spaces are considered with suitable partial orders. The respective partial order on $C^m_{nd}(\Omega)$ may depend on the nonlinear partial differential operator $T(x,D)$ in (17), while the partial order on $C^0_{nd}(\Omega)$ is the natural pointwise one at the points where the two functions compared are both continuous. The operator $T(x,D)^{\#}$ is a natural extension of the nonlinear partial differential operator $T(x,D)$ in (17) to the mentioned Dedekind order completions.

Equation (18) means that for every right hand term $f \in (C^0_{nd}(\Omega))^{\#}$ in (1), there exists a solution $u \in (C^m_{nd}(\Omega))^{\#}_T$, and as seen later, the set $(C^0_{nd}(\Omega))^{\#}$ contains many discontinuous functions beyond those piecewise discontinuous.

There is an obvious ambiguity with the piecewise smooth functions in $C^m_{nd}(\Omega)$. Indeed, given any such function u, the corresponding closed, nowhere dense set Γ cannot be defined uniquely. Therefore, it is convenient to factor out this ambiguity. For the space $C^0_{nd}(\Omega)$ which is the largest of these spaces of functions and also the range of $T(x,D)$ in (17) this is done by defining on it the equivalence relation $u \sim v$ for any two elements $u, v \in C^0_{nd}(\Omega)$, as given by

$$u \sim v \Longleftrightarrow \left[\begin{array}{l} \exists\, \Gamma \ \subset \ \Omega \text{ closed, nowhere dense:} \\ \quad \text{(i) } u, v \in C(\Omega \setminus \Gamma) \\ \quad \text{(ii) } u = v \ \text{ on } \ \Omega \setminus \Gamma \end{array} \right]. \tag{19}$$

The mentioned ambiguity is eliminated by going to the quotient space

$$\mathcal{M}^0(\Omega) = C^0_{nd}(\Omega)/\sim \tag{20}$$

The partial order on $C^0_{nd}(\Omega)$ induces a partial order on the quotient space $\mathcal{M}^0(\Omega)$, namely, for any two $\mathbf{u},\ \mathbf{v} \in \mathcal{M}^0(\Omega)$ we have

$$\mathbf{u} \le \mathbf{u} \Longleftrightarrow \left[\begin{array}{l} \exists\, u \in \mathbf{u},\, v \in \mathbf{v},\, \Gamma \subset \Omega \text{ closed, nowhere dense:} \\ \text{i) } u, v \in C(\Omega \setminus \Gamma) \\ \text{ii) } u \le v \ \text{ on } \ \Omega \setminus \Gamma \end{array} \right]. \tag{21}$$

Using similar manipulations, this time also involving the operator $T(x,D)$, the ambiguity in the domain of $T(x,D)$ in (17) is factored out, thus producing the space $\mathcal{M}^0_T(\Omega)$ with partial order which may also depend on the operator T. For details on this procedure see [13]. The equation (18) is now replaced by

$$T(x,D)^{\#}\ (\mathcal{M}^m_T(\Omega))^{\#}_T \ = \ (\mathcal{M}^0(\Omega))^{\#} \tag{22}$$

The basic regularity result in [13] is obtained by embedding $(\mathcal{M}^0(\Omega))^{\#}$ in the set of all measurable functions on Ω. Hence the solutions of (1) can be assimilated with measurable functions.

In the next section we will show that $\mathcal{M}^0(\Omega)$ can be embedded in the set $\mathbb{H}(\Omega)$ of all Hausdorff continuous functions on Ω. Since the set $\mathbb{H}(\Omega)$ is order complete it also contains the Dedekind order completion of $\mathcal{M}^0(\Omega)$. More precisely, we obtain that $\mathcal{M}^0(\Omega)$ is order isomorphic to $\mathbb{H}_{nf}(\Omega)$. Hence the solutions of (1) can be assimilated with nearly finite Hausdorff continuous functions.

4 Assimilating the solutions of nonlinear PDEs with Hausdorff continuous functions

Let $u \in C_{nd}(\Omega)$. According to (16), there exists a closed, nowhere dense set $\Gamma \subset \Omega$ such that $u \in C(\Omega \setminus \Gamma)$. Since $\Omega \setminus \Gamma$ is open and dense in Ω, we can define

$$F_0(u) = F(\Omega \setminus \Gamma, \Omega, u) \tag{23}$$

The closed, nowhere dense set Γ used in (23), is not unique. However, we can show that the value of $F(\Omega \setminus \Gamma, \Omega, u)$ does not depend on the set Γ in the sense that for every closed, nowhere dense set Γ such that $u \in C(\Omega \setminus \Gamma)$ the value of $F(\Omega \setminus \Gamma, \Omega, u)$ remains the same.

Let Γ_1 and Γ_2 be closed, nowhere dense sets such that $u \in C(\Omega \setminus \Gamma_1)$ and $u \in C(\Omega \setminus \Gamma_2)$. Then the set $\Gamma_1 \cup \Gamma_2$ is also closed and nowhere dense. According to Theorem 11 in the Appendix the functions $F(\Omega \setminus \Gamma_1, \Omega, u)$, $F(\Omega \setminus \Gamma_2, \Omega, u)$ and $F(\Omega \setminus (\Gamma_1 \cup \Gamma_2), \Omega, u)$ are all H-continuous and for every $x \in \Omega \setminus (\Gamma_1 \cup \Gamma_2)$ we have

$$F(\Omega \setminus \Gamma_1, \Omega, u)(x) = F(\Omega \setminus \Gamma_2, \Omega, u)(x) = F(\Omega \setminus (\Gamma_1 \cup \Gamma_2), \Omega, u)(x) = u(x).$$

Since $\Omega \setminus (\Gamma_1 \cup \Gamma_2)$ is dense in Ω Theorem 9 implies that

$$F(\Omega \setminus \Gamma_1, \Omega, u) = F(\Omega \setminus \Gamma_2, \Omega, u) = F(\Omega \setminus (\Gamma_1 \cup \Gamma_2), \Omega, u).$$

Therefore, the mapping

$$F_0 : C_{nd}(\Omega) \longmapsto \mathbb{A}(\Omega)$$

is unambiguously defined through (23). In analogy with (7), we call F_0 a graph completion mapping on $C_{nd}(\Omega)$. As mentioned above already it follows from Theorem 11 that for every $u \in C_{nd}(\Omega)$ we have

$$F_0(u) \in \mathbb{H}(\Omega).$$

Furthermore, if $u \in C(\Omega \setminus \Gamma)$ we have

$$F_0(u)(x) = u(x), \;\; x \in \Omega \setminus \Gamma. \tag{24}$$

The above identity shows that the values of the function $F_0(u)$ are finite on the open and dense set $\Omega \setminus \Gamma$. Hence, $F_0(u)$ is nearly finite, see Definition 1. Thus, we have

$$F_0 : C_{nd}(\Omega) \longmapsto \mathbb{H}_{nf}(\Omega) \tag{25}$$

The following theorem shows that the images of two functions in $C_{nd}(\Omega)$ under the mapping F_0 are the same if and only if these functions are equivalent with respect to the relation (19).

Theorem 5 *Let $u, v \in C_{nd}(\Omega)$. Then*

$$F_0(u) = F_0(v) \iff u \sim v$$

Proof. Implication to the left. Let Γ be closed, nowhere dense subset of Ω associated with u and v in terms of (19), that is,

$$\begin{aligned} & u, v \in C(\Omega \setminus \Gamma), \\ & u(x) = v(x),\ x \in \Omega \setminus \Gamma. \end{aligned}$$

The required equality follow from (23) where the set Γ is the one considered above. Indeed, we have

$$F_0(u) = F(\Omega \setminus \Gamma, \Omega, u) = F(\Omega \setminus \Gamma, \Omega, v) = F_0(v)$$

Implication to the right. Let us denote by Γ_1 and Γ_2 the closed, nowhere dense sets associated with the functions u and v, respectively, in terms of (16), that is, Γ_1 and Γ_2 are such that $u \in C(\Omega \setminus \Gamma_1)$ and $v \in C(\Omega \setminus \Gamma_2)$. Assume that

$$F_0(u) = F_0(v). \tag{26}$$

The functions u and v are both continuous on the set $\Omega \setminus (\Gamma_1 \cup \Gamma_2)$. Therefore, from the property (24) and the assumption (26) it follows that

$$u(x) = F_0(u)(x) = F_0(v)(x) = v(x),\ x \in \Omega \setminus (\Gamma_1 \cup \Gamma_2).$$

Since the set $\Gamma_1 \cup \Gamma_2$ is closed and nowhere dense in Ω, the above identity implies that $u \sim v$, see (19). ■

In view of (25) and Theorem 5 now we can define a mapping

$$\mathbf{F}_0 : \mathcal{M}^0(\Omega) \longmapsto \mathbb{H}_{nf}(\Omega)$$

in the following way. Let $\mathbf{u} \in \mathcal{M}^0(\Omega)$ and let $\phi \in \mathbb{H}_{nf}(\Omega)$. Then

$$\mathbf{F}_0(\mathbf{u}) = \phi \Longleftrightarrow \exists u \in \mathbf{u} : F_0(u) = \phi. \tag{27}$$

It is easy to see that the definition of $\mathbf{F}_0(\mathbf{u})$ does not depend on the particular representative u of the equivalence class $\mathbf{u}$. Indeed, if $u, h \in \mathbf{u}$ then $u \sim h$. Thus, $F_0(u) = F_0(h)$, see Theorem 5. Therefore the statement (27) can be reformulated as

$$\mathbf{F}_0(\mathbf{u}) = \phi \Longleftrightarrow \forall u \in \mathbf{u} : F_0(u) = \phi. \tag{28}$$

Theorem 6 *The mapping $\mathbf{F}_0 : \mathcal{M}^0(\Omega) \longmapsto \mathbb{H}_{nf}(\Omega)$ defined by (27) is an order isomorphic embedding with respect to the order relation (21) in $\mathcal{M}^0(\Omega)$ and the order relation (4) in $\mathbb{H}_{nf}(\Omega)$, that is, for any $\mathbf{u}$,$\mathbf{v} \in \mathcal{M}^0(\Omega)$ we have*

$$\mathbf{u} \le \mathbf{v} \Longleftrightarrow \mathbf{F}_0(\mathbf{u}) \le \mathbf{F}_0(\mathbf{v}).$$

Proof. Let $\mathbf{u}$,$\mathbf{v} \in \mathcal{M}^0(\Omega)$ and $\mathbf{u} \le \mathbf{v}$. According to (21) there exists a closed, nowhere dense set Γ in Ω and functions $u \in \mathbf{u}$, $v \in \mathbf{v}$ such that $u, v \in C(\Omega \setminus \Gamma)$ and $u(x) \le v(x)$ for all $x \in \Omega \setminus \Gamma$. Using the same set Γ in the evaluation of $F_0(u)$ and $F_0(v)$ according to (23) as well as the monotonicity of the graph completion operator, see Theorem 12, we have

$$F_0(u) = F(\Omega \setminus \Gamma, u) \le F(\Omega \setminus \Gamma, v) = F_0(v)$$

Let us assume now that $\mathbf{u}, \mathbf{v} \in \mathcal{M}^0(\Omega)$ and $\mathbf{F}_0(\mathbf{u}) \leq \mathbf{F}_0(\mathbf{v})$. From the representation (27) of $\mathbf{F}_0$ it follows that there exist $u \in \mathbf{u}$ and $v \in \mathbf{v}$ such that $F_0(u) = \mathbf{F}_0(\mathbf{u})$ and $F_0(v) = \mathbf{F}_0(\mathbf{v})$. Obviously we have

$$F_0(u) \leq F_0(v). \tag{29}$$

Since $u, v \in C_{nd}(\Omega)$ there exist closed, nowhere dense sets Γ_1 and Γ_2 such that $u \in C_{nd}(\Omega \setminus \Gamma_1)$ and $v \in C_{nd}(\Omega \setminus \Gamma_2)$. The set $\Gamma = \Gamma_1 \cup \Gamma_2$ is also closed, nowhere dense. Both functions u and v are continuous on $\Omega \setminus \Gamma$. Therefore $F_0(u)(x) = u(x)$ and $F_0(v)(x) = v(x)$ for all $x \in \Omega \setminus \Gamma$, see (24). Hence, inequality (29) implies

$$u(x) \leq v(x),\ x \in \Omega \setminus \Gamma,$$

which means that $\mathbf{u} \leq \mathbf{v}$, see (21). ■

Theorem 7 *Let $h \in \mathbb{H}_{nf}(\Omega)$. There exists a subset $\mathcal{G}$ of $\mathcal{M}^0(\Omega)$ such that $h = \sup \mathbf{F}_0(\mathcal{G})$, where $\mathbf{F}_0(\mathcal{G})$ is the range of $\mathcal{G}$ under $\mathbf{F}_0$, that is, $\mathbf{F}_0(\mathcal{G}) = \{\mathbf{F}_0(\mathbf{u}) : \mathbf{u} \in \mathcal{G}\}$.*

Proof. The set $\Gamma_{nf}(h) = \{x \in \Omega : \infty \in h(x) \text{ or } -\infty \in h(x)\}$ is closed, nowhere dense, see Definition 1, and $h \in \mathbb{H}_{ft}(\Omega \setminus \Gamma_{nf}(h))$. Then according to Theorem 14 the function h can be represented on the set $\Omega \setminus \Gamma_{nf}(h)$ as

$$h(x) = (\sup \mathcal{F})(x)\ ,\ x \in \Omega \setminus \Gamma_{nf}(h), \tag{30}$$

where

$$\mathcal{F} = \{v \in C(\Omega \setminus \Gamma_{nf}(h)) : v(x) \leq h(x)\ ,\ x \in \Omega \setminus \Gamma_{nf}(h)\}.$$

The set $\mathcal{F}$ is a subset of $C_{nd}(\Omega)$ because $\Gamma_{nf}(h)$ is closed and nowhere dense. We will show that

$$h = \sup \mathbf{F}_0(\mathcal{G})$$

where

$$\mathcal{G} = \{\mathbf{v} \in \mathcal{M}^0(\Omega) : \exists v \in \mathcal{F} : v \in \mathbf{v}\}.$$

Indeed, since all functions in $\mathcal{F}$ are continuous on $\Omega \setminus \Gamma_{nf}(h)$, for every $\mathbf{v} \in \mathcal{G}$ and $v \in \mathbf{v}$ we have, see Theorem 10,

$$v(x) = F_0(v)(x) = \mathbf{F}_0(\mathbf{v})(x)\ ,\ x \in \Omega \setminus \Gamma_{nf}(h). \tag{31}$$

Hence

$$\mathbf{F}_0(\mathbf{v})(x) = v(x) \leq h(x)\ ,\ x \in \Omega \setminus \Gamma_{nf}(h)\}.$$

Using that both $\mathbf{F}_0(\mathbf{v})$ and h are H-continuous on Ω we obtain from Theorem 9 that

$$\mathbf{F}_0(\mathbf{v})(x) \leq h(x)\ ,\ x \in \Omega\ ,\ \mathbf{v} \in \mathcal{G}.$$

Therefore, h is an upper bound of $\mathbf{F}_0(\mathcal{G})$. As a bounded subset of $\mathbb{H}_{nf}(\Omega)$ the set $\mathbf{F}_0(\mathcal{G})$ has a supremum in $\mathbb{H}_{nf}(\Omega)$, see Theorem 13. Let $g = \sup \mathbf{F}_0(\mathcal{G})$. Clearly

$$g \leq h. \tag{32}$$

Furthermore, from (31) it follows that for every $v \in \mathcal{F}$ and the respective class $\mathbf{v} \in \mathcal{G}$ containing v we have

$$v(x) = \mathbf{F}_0(\mathbf{v})(x) \leq g(x)\ ,\ x \in \Omega \setminus \Gamma_{nf}(h).$$

Hence, g is an upper bound of $\mathcal{F}$ on the set $\Omega \setminus \Gamma_{nf}(h)$ while h is the supremum of $\mathcal{F}$ on $\Omega \setminus \Gamma_{nf}(h)$. Therefore,

$$h(x) \leq g(x),\ x \in \Omega \setminus \Gamma_{nf}(h).$$

Using the H-continuity of g and h, from Theorem 9 we obtain that

$$h(x) \leq g(x),\ x \in \Omega.$$

This together with (32) shows that $h = g = \sup \mathbf{F}_0(\mathcal{G})$ which completes the proof. ■

The Theorem 7 shows that $\mathbb{H}_{nf}(\Omega)$ is the smallest Dedekind order complete subset of $\mathbb{H}(\Omega)$ which contains the image of $\mathcal{M}^0(\Omega)$ under the order isomorphic embedding $\mathbf{F}_0$. Hence it is order isomorphic to the Dedekind order completion $\mathcal{M}^0(\Omega)^{\#}$ of $\mathcal{M}^0(\Omega)$. The mapping discussed in this section are illustrated on the following diagram, $\mathbf{F}_0{}^{\#}$ denoting the order isomorphism from $\mathcal{M}^0(\Omega)^{\#}$ to $\mathbb{H}(\Omega)$.

$$\begin{tikzcd}
C^0_{nd}(\Omega) \arrow[r, "F_0^{\#}", "\text{(graph completion mapping)}"'] \arrow[d] & \mathbb{H}_{nf}(\Omega) \arrow[d, leftrightarrow] \\
\mathcal{M}^0(\Omega) = C^0_{nd}(\Omega)/\sim \arrow[r, "\mathbf{F}_0", "\text{(order isomorphic embedding)}"'] \arrow[d] & \mathbb{H}_{nf}(\Omega) \arrow[d, leftrightarrow] \\
\mathcal{M}^0(\Omega)^{\#} \arrow[r, "\mathbf{F}_0^{\#}", "\text{(order isomorphism)}"'] & \mathbb{H}_{nf}(\Omega)
\end{tikzcd}$$

The set of solutions $\mathcal{M}^m_T(\Omega))^{\#}_T$ is mapped onto the set $\mathbb{H}_{nf}(\Omega)$ of all nearly finite Hausdorff continuous functions through the composition of the mappings $T(x,D)^{\#}$ and $\mathbf{F}_0^{\#}$. Considering that both mappings are order isomorphisms, the set of solutions $\mathcal{M}^m_T(\Omega))^{\#}_T$ is order isomorphic with the set $\mathbb{H}_{nf}(\Omega)$. Hence, the solutions of (1) through the order completion method can be assimilated with nearly finite Hausdorff continuous functions.

5 Conclusion

The paper deals with the regularity of the solutions of nonlinear PDEs obtained through the order completion method. We show that these solutions can be

assimilated with Hausdorff continuous function, thus significantly improving the results in [13] with respect to the regularity properties of the solutions. The applications of the class of Haudorff continuous functions discussed here as well as in other recent publications, [1], [2], [4], show that this class may play an important role in what is typically called Real Analysis. In particular, one may note that one of the main engines behind the development of the various spaces in Real and Abstract Analysis are the partial differential equations with the need to assimilate the various types of "weak" solutions. Since the solutions of very large classes of nonlinear partial differential equations can be assimilated with nearly finite Hausdorff continuous functions, the set of these functions might be a viable alternative to some of the presently used functional spaces (e.g. $L^p(\Omega)$, Sobolev spaces) with the advantage of being both more regular and universal.

Appendix

The concept of Hausdorff continuous interval valued functions was developed first within the theory of Hausdorff approximations of real functions, see [14]. The name is derived from the fact that for a Hausdorff continuous function $f = [\underline{f}, \overline{f}]$ the Hausdorff distance between the graphs of $\underline{f}$ and $\overline{f}$ is zero. Since the Hausdorff continuous functions are in general interval valued they are also studied as a part of the Interval Analysis, see [3], [2].

The minimality condition associated with the Hausdorff continuity, see Definition 2, requires that the graph of a Hausdorff continuous function is as 'thin' as possible, that is, the function assumes proper interval values only when it is necessary to ensure that the graph of this interval function is a closed subset of $\Omega \times \overline{\mathbb{R}}$. As a result the set where a Hausdorff continuous function assumes proper interval values is small. The next theorem shows that this set is meager or a set of first Baire category, that is, a countable union of closed and nowhere dense sets.

Theorem 8 *The set $W_f = \{x \in \Omega : w(f(x)) > 0\}$ of all points where $f \in \mathbb{A}(\Omega)$ assumes proper interval values is a set of first Baire category.*

It may appear at first that the minimality condition in Definition 2 applies at each individual point x of Ω, thus, not involving neighborhoods. However, the graph completion operator F does appear in this condition. And this operator according to (7) and therefore (5) and (6) does certainly refer to neighborhoods of points in Ω, a situation typical, among others, for the concept of continuity. Hence the following property of the continuous functions is preserved.

Theorem 9 *Let f, g be H-continuous on Ω and let D be a dense subset of Ω. Then*

$$\begin{array}{llcll} \text{a)} & f(x) \leq g(x), & x \in D & \Longrightarrow & f(x) \leq g(x), \ x \in \Omega, \\ \text{b)} & f(x) = g(x), & x \in D & \Longrightarrow & f(x) = g(x), \ x \in \Omega. \end{array}$$

The following two theorems represent essential links with the usual point valued continuous functions.

Theorem 10 *Let* $f = [\underline{f}, \overline{f}]$ *be an H-continuous function on* Ω.
a) If $\underline{f}$ *or* $\overline{f}$ *is continuous at a point* $a \in \Omega$ *then* $\underline{f}(a) = \overline{f}(a)$.
b) If $\underline{f}(a) = \overline{f}(a)$ *for some* $a \in \Omega$ *then both* $\underline{f}$ *and* $\overline{f}$ *are continuous at* a.

Theorem 11 *Let* D *be a dense subset of* Ω. *If* $f \in C(D)$ *then*

$$\begin{aligned} &F(D, \Omega, f) \in \mathbb{H}(\Omega), \\ &F(D, \Omega, f)(x) = f(x),\ x \in D. \end{aligned}$$

A partial order which extends the total order on $\overline{\mathbb{R}}$ can be defined on $\mathbb{I}\,\overline{\mathbb{R}}$ in more than one way. Historically, several partial orders are associated with the set $\mathbb{I}\,\overline{\mathbb{R}}$, namely,
(i) the inclusion relation $[\underline{a}, \overline{a}] \subseteq [\underline{b}, \overline{b}] \Longleftrightarrow \underline{b} \le \underline{a} \le \overline{a} \le \overline{b}$
(ii) the "strong" partial order $[\underline{a}, \overline{a}] \preceq [\underline{b}, \overline{b}] \Longleftrightarrow \overline{a} \le \underline{b}$
(iii) the partial order defined by (3).
The use of the inclusion relation on the set $\mathbb{I}\,\overline{\mathbb{R}}$ is motivated by the applications of interval analysis to generating enclosures of solution sets. However, the role of partial orders extending the total order on $\overline{\mathbb{R}}$ has also been recognized in computing, see [7]. Both orders (ii) and (iii) are extensions of the order on $\overline{\mathbb{R}}$. The use of the order (ii) is based on the view point that inequality between intervals should imply inequality between their interiors. This approach is rather limiting since the order (ii) does not retain some essential properties of the order on $\overline{\mathbb{R}}$. For instance, a proper interval A and the interval $A + \varepsilon$ are not comparable with respect to the order (ii) when the positive real number ε is small enough. The partial order (iii) is introduced and studied by Markov, see [12], [11]. The results reported in [1] and in the present paper indicate that indeed the partial order (4) induced pointwise by (3) is an appropriate partial order to be associated with the Hausdorff continuous interval valued functions.

The monotonicity with respect to the relation inclusion was discussed in Section 2, see (9) and (10). The following theorem states the monotonicity of the Baire operators and the graph completion operator with respect to the order (4) induces in a pontwise way by the order (3).

Theorem 12 *The lower Baire operator, the upper Baire operator and the graph completion operator are all monotone increasing with respect to the order (4) on the respective domains and ranges, that is, if* D *is a dense subset of* Ω, *for every two functions* $f, g \in \mathbb{A}(D)$ *we have*

$$f(x) \le g(x),\ x \in D \implies \left\{ \begin{array}{l} I(D, \Omega, f)(x) \le I(D, \Omega, g)(x),\ x \in \Omega \\ S(D, \Omega, f)(x) \le S(D, \Omega, g)(x),\ x \in \Omega \\ F(D, \Omega, f)(x) \le F(D, \Omega, g)(x),\ x \in \Omega \end{array} \right.$$

Important property of the set $\mathbb{H}(\Omega)$ is that it is order complete. As we noted, the order completeness or the Dedekind order completeness is not a

typical property for the spaces of functions considered in Real Analysis. In this way, the class of H-continuous functions and its subclasses mentioned in the next theorem can provide solutions to open problems or improve earlier results related to order.

Theorem 13 .

(i) The set $\mathbb{H}(\Omega)$ of all H-continuous functions is order complete.

(ii) The set $\mathbb{H}_{bd}(\Omega)$ of all bounded H-continuous functions, that is,

$$\mathbb{H}_{bd}(\Omega) = \{f \in \mathbb{H}(\Omega) : \exists M \in \mathbb{R} : |f(x)| \leq M,\ x \in \Omega\}$$

is Dedekind order complete

(iii) The set $\mathbb{H}_{ft}(\Omega)$ of all finite H-continuous functions, that is,

$$\mathbb{H}_{ft}(\Omega) = \{f \in \mathbb{H}(\Omega) : |f(x)| < +\infty,\ x \in \Omega\}$$

is Dedekind order complete

(iv) The set $\mathbb{H}_{nf}(\Omega)$ of all nearly finite H-continuous functions, that is,

$$\mathbb{H}_{nf}(\Omega) = \{f \in \mathbb{H}(\Omega) : \exists D - \text{ dense subset of } \Omega : |f(x)| < +\infty,\ x \in D\}$$

is Dedekind order complete.

The resent paper [1] gives the Dedekind order completion of the space $C(X)$ of all continuous real functions on a topological space X in terms of Hausdorff continuous functions, thus improving significantly an earlier result by Dilworth, see [8]. The main result in [1] is stated below for the case when $X = \Omega$.

Theorem 14 *The set $\mathbb{H}_{ft}(\Omega)$ is a Dedekind order completion of the set $C(\Omega)$. Moreover, for every $h \in \mathbb{H}_{ft}(\Omega)$ we have*

$$h = sup\{f \in C(\Omega) : f \leq h\}$$

The proof of the theorems in this apendix can be found in [1] and [2].

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